Pergamon

Int. J. Solids Structures Vol. 31, No. 8, pp. 1099–1111, 1994
Elsevier Science Ltd
Printed in Great Britain
0020–7683/94 $6.00 + .00

A CONSTITUTIVE LAW FOR A CLASS OF TWO-PHASE MATERIALS WITH EXPERIMENTAL VERIFICATION

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(*Received* 26 *May* 1993; *in revised form* 10 *September* 1993)

Abstract—A constitutive equation is developed to describe the mechanical behaviour of a class of two-phase materials with locking-in inter-phase residual stress capability. The irreversible deformation of the materials is separated into the plastic deformation of the soft-phase and the elastic–plastic deformation of the hard-phase, which are described, respectively, with the corresponding divisions in a thermomechanically consistent constitutive model. The developed constitutive equation can describe the main characteristics of the two-phase materials, and is identical to the endochronic constitutive equation if the introduced secondary internal variables are ignored, and can also encompass the model proposed by Bower *et al.* as a special case. The validity of the present model is verified by the comparison between the theoretical and experimental results of OFHC and the rail steel BS11.

1. INTRODUCTION

One of the significant macroscopic responses of a two-phase material with locking-in inter-phase residual stress capability is that when subjected to asymmetric stress cycling, the material ratchets in the direction of mean stress, but the ratcheting rate gradually decreases and even vanishes when the cyclic number is sufficiently large. Further, if the mean stress is then removed the material will ratchet in the opposite direction although during the following cyclic process no mean stress exists and the material can virtually return to its original strain state after a sufficient number of cycles. This kind of phenomenon has not been observed in single-phase materials (Bower, 1987, 1989; Bower and Johnson, 1989, 1990).

Most previous analyses for this kind of two-phase material have employed idealized or simplified constitutive models such as those of perfect plasticity or linearly kinematic hardening. The former may overestimate the plastic deformation while the latter may exclude the possibility of a steady accumulation of plastic strain (Bower, 1989). Recently some more powerful models have been proposed. Bower (1989) and Bower and Johnson (1990) have proposed a new equation based upon Chaboche's model (1986) for the evolution of the back stress for a von Mises yield condition and the associate flow rule, which greatly improved the description for this type of material. Unfortunately, this model cannot predict the steady ratcheting which was observed recently in experiments on the rail steel BS11—a typical complex two-phase material with locking-in inter-phase residual stress capability (Peng and Ponter, 1993).

In this work a constitutive relation is proposed for a two-phase material, which is based on the assumption that the macroscopic irreversible deformation can be microscopically separated into the plastic deformation in the soft phase and the elastic–plastic deformation in the hard phase. The present constitutive equation is identical to the endochronic constitutive equation by Valanis (1980) if the secondary internal variables are ignored, and can encompass the constitutive law of Bower (1989), as a special case. The comparison of the material response of OFHC (single-phase material) and BS11 (two-phase material) under

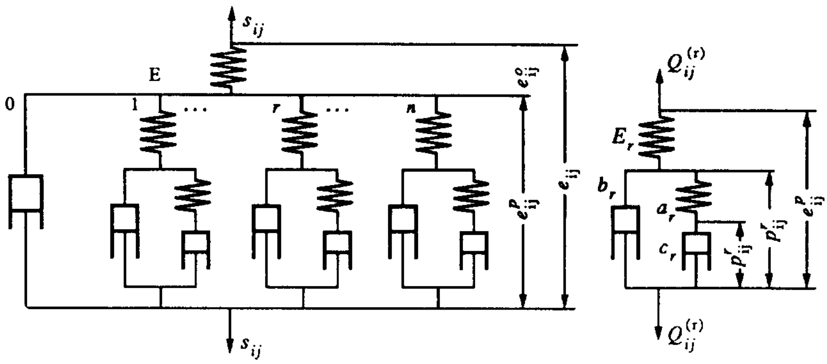

Fig. 1. Mechanical model for the developed constitutive equation.

proportional or biaxially non-proportional loading with experimental tests demonstrates the validity of this model. Compared with the model proposed by Bower (1989), the present model produces a better description for the constitutive response of the materials with locking-in inter-phase residual stress capability and can predict the non-vanishing steady ratcheting rate of the two-phase materials in unsymmetrical stress-controlled cyclic process.

2. CONSTITUTIVE EQUATION

The constitutive equation developed in this paper is restricted to initially isotropic and plastically incompressible materials under the condition of isothermal and small deformation. For simplicity of understanding, a simple mechanical model is introduced (see Fig. 1). In Fig. 1, the rth dissipative mechanism is described by the springs E_r and a_r (with stiffness E_r and a_r, respectively), plastic dashpot blocks b_r and c_r (with damping coefficients b_r and c_r, respectively). E_r is related to stochastic internal structure on the microlevel and makes no contribution to the macroscopic elastic shear modulus μ. The energy stored in the spring E_r corresponds to that stored in microstress fields determined by the respective pattern of lattice defects, for instance, dislocation. It is conjectured, from the macroscopic behaviour of the two-phase materials with locking-in inter-phase residual stress capability, that different deformation mechanisms exist in different phases during the plastic deformation process: in the soft-phase plastic deformation plays a principal role, while in the hard-phase there may exist both elastic and plastic deformation. These two mechanisms are described respectively by the two branches in Fig. 1(b), where the plastic dashpot blocks b_r describe the irreversible deformation in the soft phase, while the irreversible deformation in the hard-phase is described with spring a_r and plastic dashpot block c_r. From Fig. 1(a) and (b) we have

$$s_{ij} = Q^{(0)}_{ij} + \sum_{r=1}^{n} Q^{(r)}_{ij} \tag{1}$$

$$Q^{(r)}_{ij} = E_r(e^p_{ij} - p^{(r)}_{ij}) \tag{2}$$

with

$$e^p_{ij} = e_{ij} - e^e_{ij} = e_{ij} - \frac{s_{ij}}{2\mu}, \tag{3}$$

where e^p_{ij}, e^e_{ij} and e_{ij} represent plastic, elastic and total deviatoric strains, respectively, s_{ij} denotes the deviatoric stress, and $p^{(r)}_{ij}$ and $Q^{(r)}_{ij}$ the rth deviatoric internal variable and the corresponding generalized frictional force that satisfies the following relation

$$Q_{ij}^{(r)} = Q_{ij}^{(r)'} + Q_{ij}^{(r)''} \tag{4}$$

in which

$$Q_{ij}^{(r)'} = b_r \frac{\mathrm{d}p_{ij}^{(r)}}{\mathrm{d}z} \tag{5}$$

$$Q_{ij}^{(r)''} = c_r \frac{\mathrm{d}p_{ij}^{(r)''}}{\mathrm{d}z} \quad \text{and} \quad Q_{ij}^{(r)''} = a_r(p_{ij}^{(r)} - p_{ij}^{(r)''}) \tag{6}$$

z is the intrinsic time which is introduced to measure the inelastic strain history and $\mathrm{d}z$ is determined in terms of the distance between two adjacent inelastic strain states, $\mathrm{d}\zeta$, and the hardening function $f(z)$, as used by Wu and Yang (1983), and Wu *et al.* (1984)

$$\mathrm{d}z = \frac{\mathrm{d}\zeta}{f(z)}, \tag{7}$$

where

$$\mathrm{d}\zeta = \sqrt{\mathrm{d}e_{ij}^{p}\,\mathrm{d}e_{ij}^{p}}. \tag{8}$$

The hardening function, $f(z)$, is closely related to internal structures, which is material- and inelastic-deformation dependent.

From the employed mechanical model one can see that during a stress-controlled cyclic deformation process the soft phase suffers from pure irreversible deformation, while the hard phase has elastic and plastic deformation. If c_r is limited, the macroscopic response will tend to a steady-state of cyclic ratcheting provided there exists a mean stress. If c_r tends to infinity, the corresponding plastic dashpot block disappears, then ratcheting tends to stop when the spring a_r (or the hard phase) carries the whole mean stress so that the soft phase is only subjected to a symmetrical stress cycle (if any mean stress is still exerted on the soft phase, the soft phase will ratchet in the direction of the mean stress during cyclically plastic deformation process, which, in turn, results in the increase of the elastic deformation and the mean stress in the hard phase. This process continues until the soft phase is no longer subjected to any mean stress, and the cyclic ratcheting, therefore, stops.) When the mean stress is removed, the elastically deformed spring a_r (or the hard phase) will exert a mean stress on the soft phase. This mean stress is in the direction opposite to the originally applied external mean stress and will make the soft phase cyclically ratchet in the direction of the present mean stress, which, in turn, results in a decrease of the elastic deformation and the mean stress of a_r (or the hard phase). This process continues until the mean stress in a_r (or the hard phase) is completely released. This is just the behaviour of BS11, a typical two-phase material with locking-in inter-phase residual stress capability, reported by Bower (1989). If c_r is limited, there might exist some residual accumulated strain that cannot be completely erased during the following cycling when the mean stress has been removed, as experimentally observed (Peng and Ponter, 1993).

$Q_{ij}^{(0)}$ can be expressed in the same form as $Q_{ij}^{(r)}$. If we choose $b_0 = s_y^0$, and notice that $p_{ij}^{(0)} = e_{ij}^{p}$, then eqn (5) becomes

$$Q_{ij}^{(0)} = s_y^0 \frac{\mathrm{d}e_{ij}^{p}}{\mathrm{d}z} = s_y^0 f(z) \frac{\mathrm{d}e_{ij}^{p}}{\mathrm{d}\zeta}. \tag{9}$$

From eqns (2) and (4)–(6), one can derive the following relation (see the Appendix)

$$\mathrm{d}Q_{ij}^{(r)} = E_r\,\mathrm{d}e_{ij}^{p} + H_{ij}^{(r)}\,\mathrm{d}z, \tag{10}$$

where

$$H_{ij}^{(r)} = -\alpha_r Q_{ij}^{(r)} + \beta_r \gamma_r \Omega_{ij}^{(r)} + \beta_r E_r [e_{ij}^p - \gamma_r \eta_{ij}^{(r)}] \tag{11}$$

$$\Omega_{ij}^{(r)} = \int_0^z e^{-\gamma_r(z-z')} Q_{ij}^{(r)}(z')\,dz'$$

$$\eta_{ij}^{(r)} = \int_0^z e^{-\gamma_r(z-z')} e_{ij}^p(z')\,dz' \tag{12}$$

$$\alpha_r = \frac{E_r}{b_r} + \frac{a_r}{b_r}, \quad \beta_r = \frac{a_r}{b_r}, \quad \gamma_r = \frac{a_r}{c_r}. \tag{13}$$

It is obvious in this model that $\Sigma Q_{ij}^{(r)}$ in eqn (1) takes the part of the back stress or the centre of the yield surface, while the size of the yield surface is determined by $s_y^0 f(z)$, since from eqns (1) and (9), one has

$$s_{ij} - \sum_{r=1}^{n} Q_{ij}^{(r)} = s_y^0 f(z) \frac{de_{ij}^p}{d\zeta}. \tag{14}$$

Combining eqns (8) and (14) immediately gives

$$\left\| s_{ij} - \sum_{r=1}^{n} Q_{ij}^{(r)} \right\| = s_y^0 f(z), \tag{15}$$

where $\|A_{ij}\|$ denotes the Euclidian norm of A_{ij}. The form of the derived constitutive relation, eqns (10) and (14), is identical to that of the endochronic constitutive theory. We, therefore, introduce the following rule originally proposed by Valanis (1980) and Watanabe and Atluri (1986) to classify the material behaviour:

(i) the deformation is purely elastic if

$$\left\| s_{ij} - \sum_{r=1}^{n} Q_{ij}^{(r)} \right\| < s_y^0 f(z) \quad \text{or}$$

$$\left\| s_{ij} - \sum_{r=1}^{n} Q_{ij}^{(r)} \right\| = s_y^0 f(z) \quad \text{and} \quad \left(s_{ij} - \sum_{r=1}^{n} Q_{ij}^{(r)} \right) de_{ij} \leqslant 0. \tag{16}$$

(ii) The deformation is non-elastic if

$$\left\| s_{ij} - \sum_{r=1}^{n} Q_{ij}^{(r)} \right\| = s_y^0 f(z) \quad \text{and} \quad \left(s_{ij} - \sum_{r=1}^{n} Q_{ij}^{(r)} \right) de_{ij} > 0.$$

Although the evolution of $Q_{ij}^{(r)}$ in eqn (10) seems to be the same as the differential form of $Q_{ij}^{(r)}$ in the endochronic constitutive equation (Valanis and Fan, 1983), the former is able to describe more complicated cyclic ratcheting behaviour of the two-phase materials. In fact if a_r $(r = 1, 2, \ldots, n)$ vanish, i.e. the branches consisting of spring a_r and plastic dashpot block c_r are removed [see Fig. 1(b)], the present model becomes identical to that introduced by Fan (1987) which resulted in the endochronic constitutive equation. Further, by letting $c_r \to \infty$ then $p_{ij}^{(r)''} \to 0$, and choosing, in eqns (1), (2) and (4)–(6)

$$n = 1,\quad E_1 = \tfrac{2}{3}c,\quad f(z) = 1,\quad \frac{E_1}{b_1} = \sqrt{\tfrac{2}{3}}r_1,\quad \frac{a_1}{b_1} = \sqrt{\tfrac{2}{3}}r_2,\quad a_1 p_{ij}^{(1)} = P_{ij},\quad s_y^0 = \sqrt{2}k_e \tag{17}$$

one obtains

$$\begin{aligned}
&\tfrac{1}{2}(s_{ij}-Q_{ij})(s_{ij}-Q_{ij})-k_e^2 = 0\\
&\qquad\qquad de_{ij}^p = \frac{\sqrt{3}}{2}\frac{s_{ij}-Q_{ij}}{k_e}\,d\lambda\\
&\qquad\qquad dQ_{ij} = \tfrac{2}{3}c\,de_{ij}^p - r_1(Q_{ij}-P_{ij})\,d\lambda\\
&\qquad\qquad dP_{ij} = r_2(Q_{ij}-P_{ij})\,d\lambda.
\end{aligned} \tag{18}$$

This is just the model proposed by Bower (1989) and Bower and Johnson (1990), from which it is easily found that this model is able to account for the behaviour of the materials in which the ratcheting rate is observed to decrease with continued stress cycling. Bower has shown that in the uniaxial case the ratcheting rate is governed by the mean value of $(Q_{ij}-P_{ij})$. As eqn (18) ensures that P_{ij} follows Q_{ij} during any plastic loading, the mean value of $(Q_{ij}-P_{ij})$ decreases and tends to zero under steady cyclic loading, thus the ratcheting rate decreases gradually and tends to zero. The present model [assuming condition (17) is satisfied] gives a more distinct interpretation to the above result: during a plastic deformation process the mean values of e_{ij}^p and Q_{ij} vary simultaneously, corresponding to the given mean stress. Q_{ij} has a saturation mean value corresponding to the external mean stress, while the maximum mean value of e_{ij}^p is just determined by eqn (11) when the mean value of H_{ij} tends to zero. Thus, it is easy to find from eqn (10) that there will exist no further accumulation of plastic strain.

The present model, however, gives a more general description of the behaviour of the two-phase materials with locking-in inter-phase residual stress capability. It is reported by Peng and Ponter (1993) that for BS11 the rate of cyclic ratcheting may be greatly reduced but does not vanish even if the specimen has suffered from thousands of stress-controlled cycles until fracture. Bower's model (1989) is unable to describe this phenomenon, while the present model works well through the introduction of the hereditary integrals $\Omega_{ij}^{(r)}$ and $\eta_{ij}^{(r)}$ into the evolution of $Q_{ij}^{(r)}$.

3. INTEGRAL AND INCREMENTAL FORMS OF $Q_{ij}^{(r)}$

Although in the above section a differential form of $Q_{ij}^{(r)}$ has been derived, it was found that the use of this form as the basis for incremental computation may produce a large error if Δz is not close to zero (Fan and Peng, 1993). In order to make the computational process more exact, an integral form of $Q_{ij}^{(r)}$ is derived by integrating eqn (10) as follows:

$$Q_{ij}^{(r)} = \int_0^z e^{-\alpha_r(z-z')}\left[E_r\frac{de_{ij}^p}{dz'} + \beta_r\gamma_r\Omega_{ij}^{(r)}(z') + \beta_r E_r(e_{ij}^p(z') - \gamma_r\eta_{ij}^{(r)}(z'))\right]dz'. \tag{19}$$

In an incremental numerical process, suppose the material has suffered some plastic strain history characterized by the intrinsic time z_n, the $Q_{ij}^{(r)}$ in the subsequent increment can be calculated by the following recurrent formula

$$Q_{ij}^{(r)} = e^{-\alpha_r\Delta z}Q_{ij}^{(r)}(z_n) + \left\{E_r\frac{de_{ij}^p}{dz} + \beta_r[\gamma_r\Omega_{ij}^{(r)}(z_n) + E_r(e_{ij}^p(z_n) - \gamma_r\eta_{ij}^{(r)}(z_n))]\right\}\frac{1-e^{-\alpha_r\Delta z}}{\alpha_r}, \tag{20}$$

where $\Delta z = z - z_n$. By setting

$$\Delta Q_{ij}^{(r)} = Q_{ij}^{(r)} - Q_{ij}^{(r)}(z_n). \tag{21}$$

One can derive the following incremental formula from eqn (20)

$$\Delta Q_{ij}^{(r)} = A_r \Delta e_{ij}^p + B_{ij}^{(r)} \Delta z \tag{22}$$

in which

$$\begin{aligned} A_r &= \kappa_r E_r \\ B_{ij}^{(r)} &= \kappa_r [\beta_r \gamma_r \Omega_{ij}^{(r)}(z_n) + \beta_r E_r (e_{ij}^p(z_n) - \gamma_r \eta_{ij}^{(r)}(z_n)) - \alpha_r Q_{ij}^{(r)}(z_n)] \\ k_r &= \frac{1-\mathrm{e}^{-\alpha_r \Delta z}}{\alpha_r \Delta z}. \end{aligned} \tag{23}$$

Obviously eqn (22) returns to (10) if Δz tends to zero. However eqn (22) gives a more precise description than the incremental formula directly extended from eqn (10) even though the increment is very large. The numerical result depends strongly on the value of κ_r which is closely related to Δz (Fan and Peng, 1991), so one should be very careful when applying eqn (10) in a numerical process.

4. THE INCREMENTAL CONSTITUTIVE EQUATION

With eqns (8) and (22), as well as the elastic relation

$$s_{ij} = 2\mu(e_{ij} - e_{ij}^p) \tag{24}$$

the following incremental constitutive relation can be derived in the way similar to that proposed by Watanabe and Atluri (1986)

$$\Delta s_{ij} = 2\mu\left[\Delta e_{ij} - \frac{(s_{ij}-r_{ij})(s_{kl}-r_{kl})}{C(s_y^0)^2 f^2(z)} \Gamma \Delta e_{kl}\right] \tag{25}$$

in which

$$\begin{aligned} r_{ij} &= \sum_{r=1}^{n} Q_{ij}^{(r)} \\ C &= 1 + \sum_{r=1}^{n} A_r + \frac{(s_{ij}-r_{ij})h_{ij}}{s_y^0 f^2(z)} + \frac{s_y^0 f(z)}{2\mu}\frac{\mathrm{d}f(z)}{\mathrm{d}z} \\ h_{ij} &= \sum_{r=1}^{n} \frac{B_{ij}^{(r)}}{2\mu} \end{aligned} \tag{26}$$

the parameter Γ is introduced to assess whether the deformation is purely elastic or not, and the determination of Γ follows the rule we mentioned above, i.e.

$$\Gamma = \begin{cases} 1, & \text{if } \|s_{ij}-r_{ij}\| = s_y^0 f(z) \quad \text{and} \quad (s_{ij}-r_{ij})\Delta e_{ij} > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

Equation (25) is a three-dimensional incremental constitutive equation that can be used in various cases including finite element analysis. If the a_r ($r = 1, 2, \ldots, n$) vanish, the present incremental constitutive equation eqn (25) becomes identical to that proposed by Watanabe and Atluri (1986) for the endochronic constitutive equation, which has been shown by Valanis (1980) and Watanabe and Atluri (1986) to be able to encompass many other constitutive equations as special cases.

5. DETERMINATION OF THE MATERIAL CONSTANTS AND PARAMETERS

The evolution of the hardening function $f(r)$ is chosen the same form as used by Fan and Peng (1991) as follows:

$$\frac{\mathrm{d}f(z)}{\mathrm{d}z} = \beta(c-f(z)) \quad \text{with} \quad f(0) = 1, \tag{28}$$

where β is related to the rate of hardening and c is the saturated value of $f(z)$ related to the given loading condition, for instance, plastic strain amplitude and non-proportionality (Fan and Peng, 1991). If c is constant, one can get the following explicit expression

$$f(z) = c-(c-1)\,\mathrm{e}^{-\beta z} \tag{29}$$

which is the form used by Wu and Yang (1983) and Wu *et al.* (1984).

The material constants and parameters appearing in this model are as follows: μ is the elastic shear modulus; c is the saturated value of the hardening function; β is the rate of hardening; E_r and α_r $(r = 1,2,\ldots,n)$ are constants corresponding to back stress; s_y^0 is the deviatoric initial yield stress; and β_r and γ_r $(r = 1,2,\ldots,n)$ are constants related to cyclic ratcheting. β_r $(r = 1,2,\ldots,n)$ vanish for single-phase materials and γ_r $(r = 1,2,\ldots,n)$ vanish if cyclic ratcheting may cease to develop. Approximate analysis shows that for simplicity, it is reasonable to choose

$$\beta_r = \beta_0\alpha_r, \qquad \gamma_r = \gamma_0, \quad r = 1,2,\ldots,n. \tag{30}$$

If the material is subjected to a symmetrically stress- (or strain)-controlled cycling, the mean value of e_{ij}^p, Ω_{ij} and η_{ij} are zero and compared with the first term in the brackets on the right-hand side of eqn (19), the other terms in the brackets are small enough to be neglected. By noticing that in the uniaxially tensile case

$$\mathrm{d}s_{11} = \tfrac{2}{3}\sigma, \quad \mathrm{d}s_{22} = \mathrm{d}s_{33} = -\tfrac{1}{3}\mathrm{d}\sigma, \quad \mathrm{d}e_{11}^p = \mathrm{d}\varepsilon^p, \quad \mathrm{d}e_{22}^p = \mathrm{d}e_{33}^p = -\tfrac{1}{2}\mathrm{d}\varepsilon^p \tag{31}$$

we then have

$$\begin{aligned} \sigma &= \sum_{r=0}^{n} \sigma^{(r)} \\ \sigma^{(0)} &= \sqrt{\tfrac{3}{2}}\sigma_y^0 \frac{\mathrm{d}\varepsilon^p}{\mathrm{d}z} \\ \sigma^{(r)} &= \frac{3}{2}\int_0^z E_r \mathrm{e}^{-\alpha_r(z-z')}\frac{\mathrm{d}\varepsilon^p}{\mathrm{d}z'}\mathrm{d}z', \quad r = 1,2,\ldots,n. \end{aligned} \tag{32}$$

With the approach proposed by Watanabe (1986), E_r, α_r $(r = 1,\ldots,n)$, s_y^0, c and β can be determined. Then using a cyclic process with the existence of mean stress, β_0 and γ_0 are determined to make the described ratcheting best fit the experimental data.

The determined material constants for rail steel BS11 and hard-drawn OFHC (Bower, 1989) are listed in Table 1, where $n = 2$ has been chosen.

Table 1. Material constants

Material	μ, s_y^0 (MPa)	$\alpha_{1,2}$	$C_{1,2}$	β_0	γ_0	c, β
OFHC	47,700, 12.4	600, 150	0.20, 0.69	0	0	1.08, 250
BS11	80,000, 21.6	180, 3.2	0.17, 0.23	0.033	0.036	1.56, 120

$C_r = E_r/(2\mu)$.

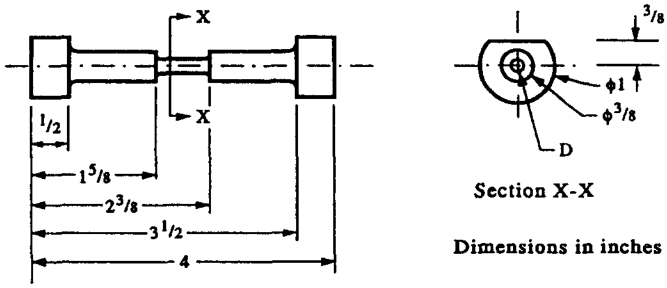

Fig. 2. Geometry of the specimen.

6. ANALYSIS AND EXPERIMENTAL VERIFICATION

Figure 2 shows the geometry of the specimen used by Bower in tension/compression and/or torsion experiments (Bower, 1989). The axial displacement is recorded by using a displacement transducer mounted between grips, and the twist is similarly measured by a transducer attached to the rotating grip (Bower, 1989). In the calculation it is assumed that plastic deformation only occurs in part of the diameter D, and part of the 3/8 inch diameter of the specimen suffers purely elastic deformation which is also taken into consideration. The responses of the specimens are calculated numerically. Because the stress caused by torsional load is not homogeneous, the cross-section with diameter D is separated into N parts along the radius and the torque increment can be approximately calculated by

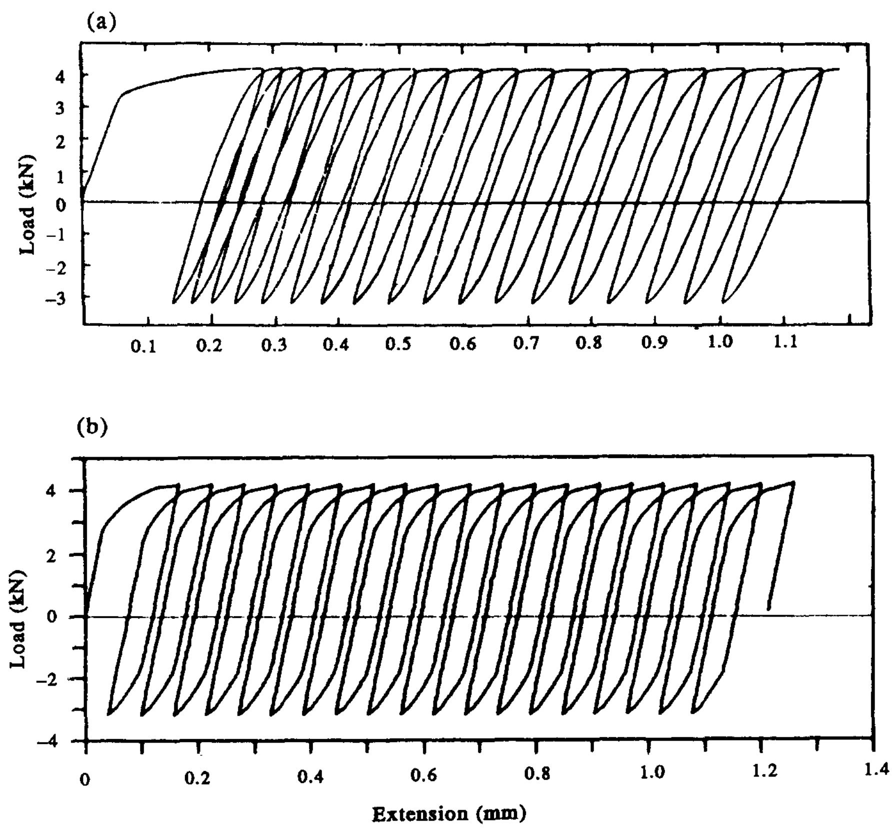

Fig. 3. Response of the OFHC specimen under tensile stress cycling. (a) Experimental and (b) theoretical.

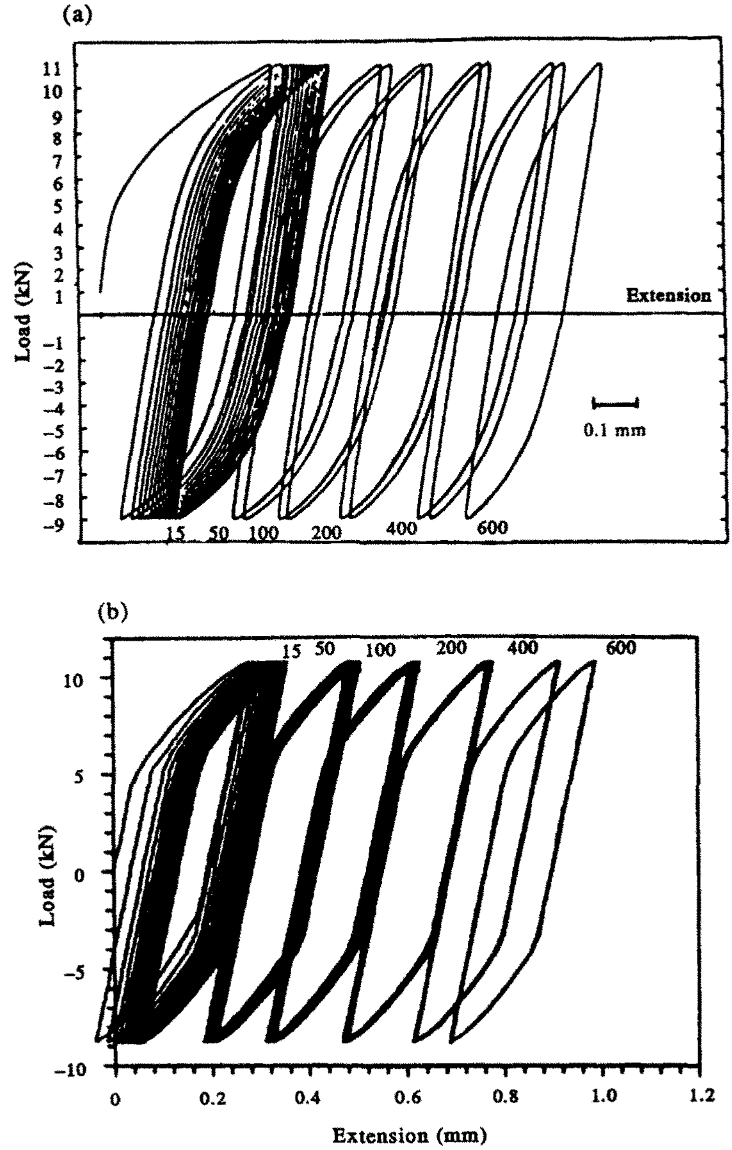

Fig. 4. Response of the BS11 specimen under tensile stress cycling. (a) Experimental and (b) theoretical.

$$\Delta M = \sum_{i=1}^{N} 2\pi (r_i')^2 \Delta\tau(r_i')\Delta r_i, \tag{33}$$

where $\Delta r_i = r_i - r_{i-1}$, $r_i' = (r_i + r_{i-1})/2$ and $r_0 = 0$. If the specimen is subjected to axial and shear deformation, then from eqn (25) we have

$$\begin{aligned} \Delta s_{11} &= 2\mu[\Delta e_{11} - \Delta a(s_{11} - r_{11})\Gamma] \\ \Delta s_{12} &= 2\mu[\Delta e_{12} - \Delta a(s_{12} - r_{12})\Gamma], \end{aligned} \tag{34}$$

where Γ is determined by eqn (27) and

$$\begin{aligned} \Delta a &= \frac{1.5(s_{11} - r_{11})\Delta e_{11} + 2(s_{12} - r_{12})\Delta e_{12}}{C[s_y^0 f(z)]^2} \\ \Delta e_{12} &= \tfrac{1}{2} r \Delta\phi. \end{aligned} \tag{35}$$

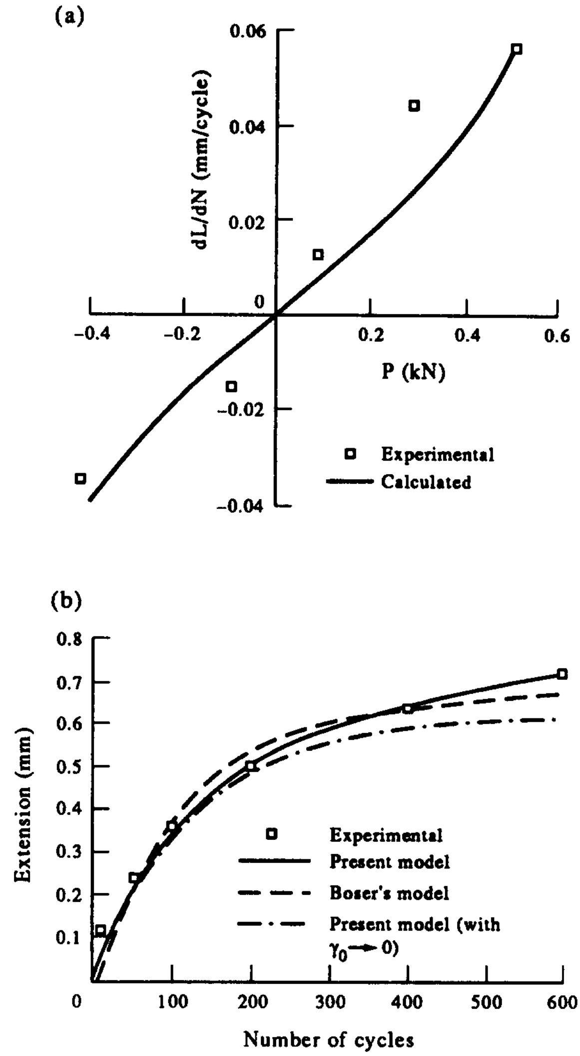

Fig. 5. (a) Variation of $\Delta l/\Delta N$ vs P_m of the OFHC specimen during tensile stress cycling. (b) Accumulated elongation vs the number of tensile stress cycle loads.

Suppose the plane assumption is appropriate, we then have the relation between incremental torque ΔM and incremental twist per unit length $\Delta\phi$, provided the cyclic process is axial strain and torque controlled

$$\Delta\phi = \frac{\dfrac{\Delta M}{2\pi\mu} + \sum\limits_{i=1}^{N} 3(r_i')^2 \Delta r_i [a\Gamma(s_{11}-r_{11})(s_{12}-r_{12})\Delta e_{11}]_{r=r_i'}}{\sum\limits_{i=1}^{N} (r_i')^3 \Delta r_i [1-2a\Gamma(s_{12}-r_{12})^2]_{r=r_i'}} \tag{36}$$

The calculated cyclically tensional load–extension curve of the OFHC specimen is shown in Fig. 3(b), where $P_a = 3.75$ kN, $P_m = 0.5$ kN and $D = 4.76$ mm, which is in good agreement with the experimental result shown in Fig. 3(a) (Bower, 1989) except for the first half cycle, where the error can mainly be attributed to the fact that the material constants obtained in the cyclic process do not work well in the case of monotonic loading. It is found from Fig. 3 that the ratcheting rate varies in the first few cycles and quickly tends to a constant in a cyclic process, which is the characteristic of the single-phase material. Figure 4(b) shows the extension of a BS11 specimen with diameter $D = 4.76$ mm and subjected to

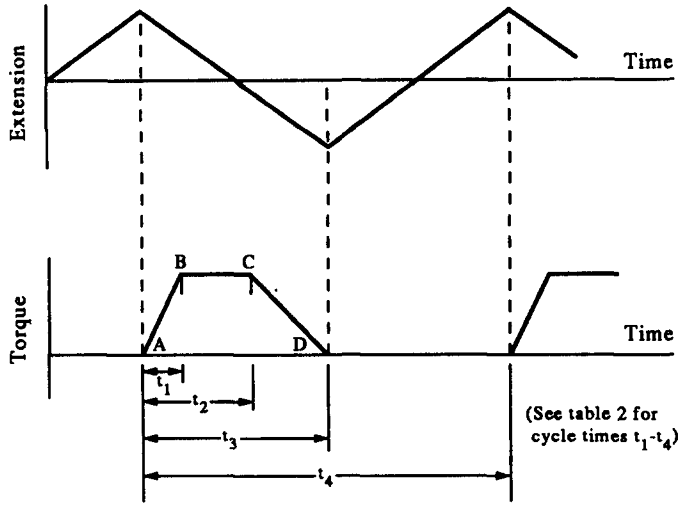

Fig. 6. Biaxial non-proportional path.

cyclic axial load with the mean value $P_m = 1$ kN and the amplitude $P_a = 10$ kN. Compared with the experimental result shown in Fig. 4(a), it can be observed that although the shape of the load–extension loops is not accurately predicted, the theoretical variation of ratcheting is very close to that measured experimentally. The decrease of the ratcheting rate of BS11 is well described by the present model. Figure 5(a) shows the calculated and experimental steady ratcheting rates of OFHC specimen subjected to a fixed load amplitude $P_a = 3.75$ kN and varying mean load P_m. Figure 5(b) shows the ratcheting elongation against cyclic number of the test of Fig. 4 and it is found that the present model fits the experimental result better than Bower's model (1989), for example at cycle 600, the present model still works well while Bower's model (1989) underestimates the cyclic ratcheting rate and the accumulated elongation. In Fig. 5(b) the dotted curve corresponds to the result calculated by the present model when the secondary variables are ignored (i.e. $\gamma_0 \to 0$), and it is seen that the tendency of this result is similar to that given by Bower's model.

A biaxial nonproportional path is shown in Fig. 6, in which the specimen is displacement-controlled in tensile direction and torque-controlled in shear direction (Bower, 1989). The governing parameters are listed in Table 2. Figure 7(b) shows the theoretical response of the OFHC specimen, which is very similar to the experimental result shown in Fig. 7(a). It is seen that the twist accumulates in the direction of mean torque and the rate quickly tends to a constant with the increase of cyclic number. This behaviour is quite different from that of BS11 specimen, for which the ratcheting rate significantly decreases during the cyclic process. The experimental extension–twist relation of the BS11 specimen (subjected to the non-proportional cycling shown in Fig. 6) and the theoretical correlation are shown in Fig. 8(a) and (b), respectively. It is seen that they are in reasonable agreement.

7. CONCLUSION

A constitutive model is developed for a class of two-phase materials with locking-in inter-phase residual stress capability, in which the macroscopic plastic deformation of the

Table 2. Summary of biaxial tests

Material	Diameter (mm)	Extension (mm)	Torque (N m)	t_1	t_2	t_3	t_4
OFHC	6.35	± 0.090	7.08	4.7	13.2	26.4	52.8
BS11	4.76	± 0.095	7.20	4.8	13.0	28.5	57.0

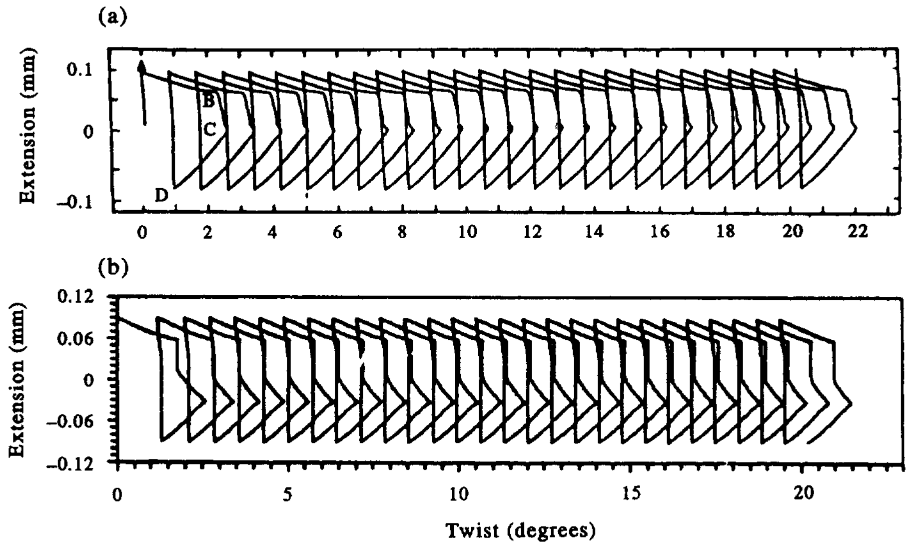

Fig. 7. Response of the OFHC specimen to non-proportional test. (a) Experimental and (b) calculated.

materials subjected to cyclic loading is microscopically separated into the plastic deformation in the soft phase and the elastic–plastic deformation in the hard phase. The developed model can include both the endochronic constitutive equation and Bower's model for the two-phase rail steel BS11 as special cases.

The integral and the incremental forms of the proposed constitutive model are also proposed for the development of the corresponding numerical algorithm.

The application to the description of the constitutive response of OFHC and BS11 subjected to proportional or non-proportional cyclic loading shows the validity of the proposed model.

Acknowledgements—The authors gratefully acknowledge the support of the British Council of Great Britain through an academic exchange agreement between the University of Leicester and Chongqing University. Thanks are also due to both universities for actively supporting this academic link.

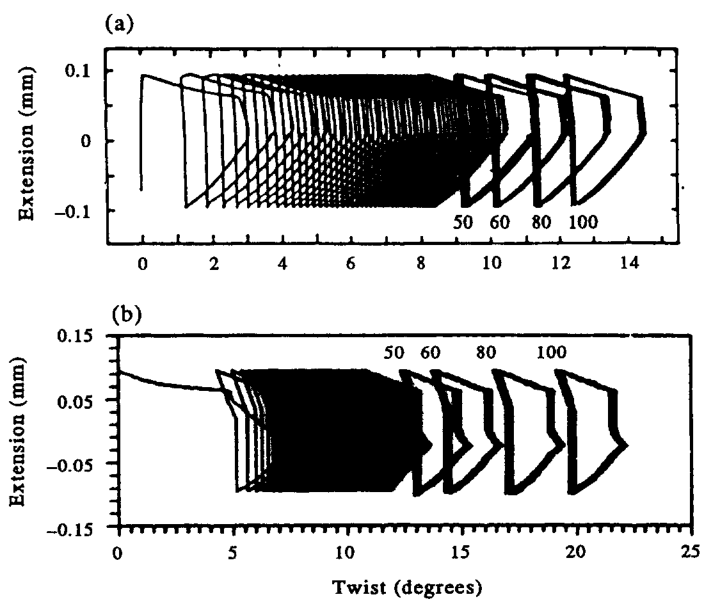

Fig. 8. Response of the BS11 specimen to non-proportional test. (a) Experimental and (b) calculated.

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APPENDIX

The following are the Laplace transformations of eqns (2) and (4)–(6)

$$
\begin{aligned}
\hat{Q}_{ij}^{(r)} &= E_r(\hat{e}_{ij}^p - \hat{p}_{ij}^{(r)}) \\
\hat{Q}_{ij}^{(r)} &= \hat{Q}_{ij}^{(r)'} + \hat{Q}_{ij}^{(r)''} \\
\hat{Q}_{ij}^{(r)'} &= b_r p \hat{p}_{ij}^{(r)} \\
\hat{Q}_{ij}^{(r)''} &= c_r p \hat{p}_{ij}^{(r)''} \\
\hat{Q}_{ij}^{(r)''} &= a_r(\hat{p}_{ij}^{(r)} - \hat{p}_{ij}^{(r)''}).
\end{aligned}
\tag{A1}
$$

One can get the following relation from the equations in (A1)

$$
b_r p \hat{Q}_{ij}^{(r)} + \frac{a_r c_r}{c_r p + a_r} \hat{Q}_{ij}^{(r)} + E_r \hat{Q}_{ij}^{(r)} = b_r E_r p \hat{e}_{ij}^p + \frac{a_r c_r p}{c_r p + a_r} E_r \hat{e}_{ij}^p. \tag{A2}
$$

Equation (A2) can also be expressed as

$$
p\hat{Q}_{ij}^{(r)} + \left(\frac{E_r}{b_r} + \frac{a_r}{b_r}\right)\hat{Q}_{ij}^{(r)} - \frac{a_r}{b_r}\frac{1}{1+\dfrac{c_r}{a_r}p}\hat{Q}_{ij}^{(r)} = E_r p \hat{e}_{ij}^p + \frac{a_r}{b_r}E_r \hat{e}_{ij}^p - \frac{a_r}{b_r}\frac{1}{1+\dfrac{c_r}{a_r}p}E_r \hat{e}_{ij}^p. \tag{A3}
$$

The inverse Laplace transformation of the above equation yields eqns (10)–(12).